3964. Proposed by George Apostolopoulos.

Let P be an arbitrary point inside a triangle ABC. Let a,b and c be the distances from P to the sides BC,AC and AB, respectively. Prove that

$$\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^4}{\sin^4 A+\sin^4 B+\sin^4 C} \le 12R^2,$$

where R denotes the circumradius of ABC. When does the equality occur?

Solution by Arkady Alt , San Jose ,California, USA.

For representation of solution we will use common and essential notations for sidelengths $a := BC, b := CA, c := AB$ and for distances from P to the sides BC,AC and AB respectively $x,y,z.$ So, original inequality becomes

(**1**) $$\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^4}{\sin^4 A+\sin^4 B+\sin^4 C} \le 12R^2 \Leftrightarrow \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^4}{a^4+b^4+c^4} \le \frac{3}{4R^2}.$$

Let F be area of the triangle. Then $ax+by+cz=2F$ and applying Cauchy Inequality to triples $\left(\sqrt{ax},\sqrt{by},\sqrt{cz}\right)$ and $\left(\frac{1}{\sqrt{a}},\frac{1}{\sqrt{b}},\frac{1}{\sqrt{c}}\right)$ we obtain

$$\left(ax+by+cz\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge \left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2 \Leftrightarrow$$

$$\frac{2F(ab+bc+ca)}{abc} \ge \left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2 \Leftrightarrow$$

$$\frac{2F}{4FR} \ge \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{ab+bc+ca} \Leftrightarrow \frac{1}{2R} \ge \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{ab+bc+ca}.$$

And since $a^2+b^2+c^2 \ge ab+bc+ca$ we obtain $\frac{1}{2R} \ge \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{a^2+b^2+c^2} \Leftrightarrow$

$$\frac{1}{4R^2} \ge \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^4}{(a^2+b^2+c^2)^2}.$$

Noting that $(a^2+b^2+c^2)^2 \le 3(a^4+b^4+c^4) \Leftrightarrow \frac{1}{(a^2+b^2+c^2)^2} \ge \frac{1}{3(a^4+b^4+c^4)}$

we finally get $\frac{3}{4R^2} \ge \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^4}{a^4+b^4+c^4}.$